POLYNOMIAL MODELS FOR INTEREST RATES AND STOCHASTIC VOLATILITY

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Abstract. This essay discusses a class of arbitrage-free interest rate models that has tractable bond price. The resulting bond price depends on some suitable factor process in a polynomial way. We also extend this class of polynomial model to the stochastic volatility case and they can produce tractable price for power options. The main result of this essay is that we have established the necessary and the sufficient conditions for these models to be arbitrage free.

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1. INTRODUCTION AND NOTATIONS

In this document, we denote r_t as the spot interest rate, $P_t(T)$ as the time t price of Zero Coupon Bond (ZCB) with maturity date $T > t$, $\{S_t\}_{t\geq 0}$ as the price process of the underlying asset, $C_t(T,K)$ as the time t price of call option with maturity T and strike K and $P_t(T,\theta)$ as the time t price of power options that pays S_T^{θ} $(0 < \theta < 1)$ at maturity date $T > t$. We also assume in this article that the filtration $\mathscr{F}$ is generated by some multidimensional Brownian motion $W = (W^1, \dots, W^m)$. For no arbitrage reason, we assume that under the EMM measure, the stock price process S_t follows:

$$dS_t = S_t\sigma_t.dW_t$$

Where the multi-dimensional process σ_t denotes the volatility. We also denote Z_t as the scalor factor process that is the solution of some suitable SDE and we are going to construct the models based on this factor Z_t.

Remark 1.1. *The fact that the factor process Z_t is a scalor process will simplify the notation and calculation quite a lot. In general, when Z_t is multi-dimensional, we still have similar results but the notation and calculation will be much complicated.*

There has been various famous class of models that produce tractable ZCB prices $P_t(T)$ and power option prices $P_t(t,\theta)$. Most of them takes the form of exponential polynomial type. For example, in the case of interest rate case, we can model the ZCB price process as:

$$P_t(T) = e^{-\int_0^{T-t} r_t(x)dx}$$

The above formula encodes the price of ZCB $P_t(T)$ by process $r_t(x)$ which is known as the forward rate and satisfy the boundary condition

$$P_T(T) = 1 \quad \forall T$$

The famous CIR [2] and Vasicek [8] model will produce the forward rate in the form

$$r_t(x) = A(x)Z_t + B(x)$$

where A, B are some deterministic functions. In the above examples forward rates depends on factor process Z_t linearly, and for quadratic type forward rates, we can see examples in Leippold and Wu [5]. Surprisingly, under some conditions, these are the only examples that produce no

arbitrage models. As Filipovic [3] points out that the maximum degree that the forward rate can have is 2, i.e. quadratic.

In the case of stock market, instead of encoding the call option price process $C_t(T, K)$, which is the natural choice, we choose to encode the power option price process $P_t(T, \theta)$. The reason behind this is that sometimes we may fail to have a closed form solution to the call price $C_t(T, K)$ but we may have closed form power option prices $P_t(T, \theta)$ [1].

Remark 1.2. *Notice that knowing the process $\{P_t(T, \theta)\}$ for all $T > 0, 0 < \theta < 1$ is equivalent to specify the conditional distribution of S_T given $\mathscr{F}_t$. Therefore it is equivalent to specify the process $\{C_t(T, K)\}$ since under EMM*

$$C_t(T, K) = \mathbb{E}[(S_T - K)^+ | \mathscr{F}_t].$$

More precisely, we have the following identities:

$$S^\theta = \theta(1-\theta) \int_0^\infty (S \wedge K) K^{\theta-2} dK \qquad \text{for all } S \geq 0, 0 < \theta < 1$$

The above idendity shows that a power option with payout S_T^θ can be replicted by a portfolio of options with payouts $S_T \wedge K = S_T - (S_T - K)^+$, and hence by portfolio of call options and the stock itself. Similarly:

$$S \wedge K = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S^{\theta+ix} \frac{K^{1-\theta-ix}}{(x - i\theta)(x + i(1-\theta))} dx \qquad \text{for all } 0 < \theta < 1$$

and hence options with payoff $S_T \wedge K$ can be replicated by generalised power options with payouts $S_T^\theta \cos(x \log S_T)$ and $S_T^\theta \sin(x \log S_T)$

Here we model the power option prices processes $P_t(T, \theta)$ as:

$$P_t(T, \theta) = S_t^\theta e^{-\frac{1}{2}\theta(1-\theta)\int_0^{T-t} f_t(x,\theta)dx}$$

Therefore we encode the process $P_t(T, \theta)$ by the process $f_t(x, \theta)$ which we called forward variance. Notice that the above re-parametrisation satisfies $P_T(T, \theta) = S_T^\theta$ for all T, θ. Similar to the interest rate case, we have examples that the forward variance $f_t(x, \theta)$ is a linear or quadratic function on factor process Z_t.

Example 1 (Heston's Model. [4])**.** *The model set up is*

$$dS_t = S_t \sqrt{Z_t} dW_t^S$$

$$dZ_t = (\bar{Z} - Z_t)dt + \sqrt{Z_t}dW_t^1$$

Where W^S and W^1 are two Brownian motions with correlation ρ and $\bar{Z}$ is some constant. As a result, the forward variance $f_t(x,\theta)$ takes the form that:

$$f_t(x,\theta) = A(x,\theta)Z_t + B(x,\theta)$$

for some deterministic functions A, B. Hence a linear example.

Example 2 (Stein-Stein Model. [7]). *With the same notation as the last example, the model set up is*

$$dS_t = S_t Z_t dW_t^S$$
$$dZ_t = \gamma dW_t^1 + (\alpha + \beta Z_t)dt$$

with some constant α, β, γ. The forward variance is given by

$$f_t(x,\theta) = Q_2(x,\theta)Z_t^2 + Q_1(x,\theta)Z_t + Q_0(x,\theta)$$

For some deterministic functions Q_i. Hence a quadratic example.

Similar as interest rate case, Michael and I have shown that under some constraints, the maximum degree that forward variance can have is up to 2. i.e. quadratic (See appendix A). Therefore together with a time dependent version of Black-Scholes Model (corresponds to the case $f_t(x,\theta) = A(x,\theta)$. degree 0 example), we have seen all the no arbitrage models.

So far, all the tractable models introduced above have a common form: the price process $P_t(T)$ or $P_t(T,\theta)$ is of the form of exponential polynomial. In the rest of this paper, we are going to investigate a new class of tractable models that takes the form of polynomial instead of exponential polynomial.

The outline of the paper is as follows. In section 2, we introduce the polynomial interest rate model and provide a sufficient and a necessary condition for these model to admits no arbitrage. In section 3, we do a similar treatment to the stock market case. And finally in section 4, we give some further analysis and provide some examples.

2. POLYNOMIAL INTEREST RATE MODELS

2.1. **Model Setup.** In this section, we are going to discuss a class of interest rate models that takes the form:

$$P_t(T) = \sum_{i=0}^{n} g_i(T-t)\phi_i(Z_t)$$

Where g_i, ϕ_i are some deterministic functions and the scaler factor process Z_t follows:

$$dZ_t = a(Z_t)dt + b(Z_t)dW_t$$

Remark 2.1. *Notice that if we set $\phi_i(z) = z^i$, the resulting $P_t(T)$ will be in polynomial form.*

The boundary condition $P_T(T) = 1$ can be made to hold true by setting $g_0(0) = 1$, $g_i(0) = 0 \quad i = 1, 2, \ldots, n$ and $\phi_0(z) = 1$. Here follows, we will always use this type of boundary condition.

Remark 2.2. *We are safe to consider only boundary conditions of the type above. If there exists some functions $\phi_i(z)$ that is independent of z, then we can absorb the constant into the coefficient function $g_i(x)$ and assume $\phi_i(z) = 1$. If however, all $\phi_i(z)$ depends on z, we then have:*

$$\sum_{i=0}^{n} g_i(0)\phi_i(z) = 1$$

$$\phi_0(z) = \frac{1 - \sum_{i=1}^{n} g_i(0)\phi_i(z)}{g_0(z)}$$

Hence our polynomial model reads:

$$\begin{aligned} P_t(T) = \sum_{i=0}^{n} g_i(T-t)\phi_i(Z_t) &= g_0(T-t)\phi_0(Z_t) + \sum_{i=1}^{n} g_i(T-t)\phi_i(Z_t) \\ &= \frac{g_0(T-t)}{g_0(0)} + \sum_{i=1}^{n} g_i(T-t)\left(1 - \frac{g_0(T-t)}{g_0(0)}\right)\phi_i(Z_t) \end{aligned}$$

We then can re-define:

$$G_0(T-t) := \frac{g_0(T-t)}{g_0(0)}$$

$$G_i(T-t) := g_i(T-t)\left(1 - \frac{g_0(T-t)}{g_0(0)}\right)$$

This re-parametrization will satisfy the above type of boundary condition.

We also assume that the spot interest rate process r_t also depends on the factor process Z_t. i.e. $r_t = R(Z_t)$ for some deterministic function R. Hence the discounted ZCB price process $\tilde{P}_t(T)$ can

be expressed as:

$$\begin{aligned}\tilde{P}_t(T) &= \exp\left(-\int_0^t r_s ds\right) P_t(T) \\ &= \exp\left(-\int_0^t R(Z_s) ds\right) \sum_{i=0}^n g_i(T-t)\phi_i(Z_t)\end{aligned}$$

2.2. **No Arbitrage Condition.** For consistency, we need:

$$\text{(A) : The process } \tilde{P}_t(T) \text{ is local martingale under some EMM } \mathbb{Q} \quad \forall T$$

For no arbitrage, we have:

Theorem 2.1. *Condition (A) holds if and only if*

$$\sum_{i=0}^n \dot{g}_i(x)\phi_i(z) = \sum_{i=0}^n g_i(x)\left(a(z)\frac{d\phi_i(z)}{dz} + \frac{b^2(z)}{2}\frac{d^2\phi_i(z)}{dz^2} - R(z)\phi_i(z)\right)$$

holds for all (x,z).

Proof:

By Ito's formula, we have:

$$\begin{aligned}d\tilde{P}_t(T) &= \exp\left(-\int_0^t R(Z_s)ds\right)(-R(Z_t)P_t(T)dt + dP_t(T)) \\ dP_t(T) &= \sum_{i=0}^n -\dot{g}_i\phi_i dt + g_i\frac{d\phi_i}{dz}dZ_t + \frac{g_i}{2}\frac{d^2\phi_i}{dz^2}d[Z]_t\end{aligned}$$

Setting the coefficient of dt to be 0 and we get the desired identity. $\square$

Before moving on, we are now going to define some notation.

$$A_i(z) := a(z)\frac{d\phi_i(z)}{dz} + \frac{b^2(z)}{2}\frac{d^2\phi_i(z)}{dz^2} - R(z)\phi_i(z)$$

Let F_k be the family of polynomials of degree at most k. That is:

$$F_k = \left\{\sum_{i=0}^k f_i z^i | f_i \in \mathbb{R}\right\}$$

It follows immediately that F_k is closed under addition and subtraction and $F_k \subset F_{k+1}$.

Theorem 2.2. *Suppose the functions* g_i, ϕ_i *are given, if further suppose the functions* $g_i(\cdot)$ *are linearly independent, we have that* $A_i(z)$ *are uniquely determined and can be expressed as linear combinations of* $\phi_i(z)$*. Moreover in the polynomial interest rate model when* $\phi_i(z) = z^i$*, we have*

$A_i(z) \in F_n$ for $i = 0, 1, \ldots, n$.

Proof:

The identity of Thm 2.1 may be rewrite as:

$$(*) \qquad \sum_{i=0}^{n} \dot{g}_i(x)\phi_i(z) = \sum_{i=0}^{n} g_i(x)A_i(z)$$

Fix any z, There are $n+1$ unknowns, namely $A_i(z)$. We can take $n+1$ sample points x_j, $j = 0, 1, \ldots, n$, such that we can rewrite (*) in the matrix form:

$$M\underline{x} = \underline{c}$$

Where $\underline{x} = (A_0(z), \ldots, A_n(z))^T$, $\underline{c} = \left(\sum_{i=0}^{n} \dot{g}_i(x_0)\phi_i(z), \ldots, \sum_{i=0}^{n} \dot{g}_i(x_n)\phi_i(z)\right)^T$ and the matrix has component $M_{ij} = g_j(x_i)$. Further by independency of fuctions g_i, we may take M to be invertible and hence we have a unique solution $\underline{x} = M^{-1}\underline{c}$. Notice that the components of $\underline{c}$ are linear combinations of $\phi_i(z)$, so does the solution $M^{-1}\underline{c}$. Finally since z was fixed arbitrarily, we get the required result. □

2.3. **Necessary Conditions.**

Theorem 2.3 (Necessity)**.** *In polynomial models, if the coefficient functions $g_i(\cdot)$ are linearly independent for $i = 0, 1, \ldots, n$ then we have $a(z), b^2(z), R(z)$ are all polynomials and $R(z) \in F_2$, $a(z) \in F_3$, $b^2(z) \in F_4$. Further if $a(z) = \sum_{i=0}^{3} a_i z^i$, $R(z) = \sum_{i=0}^{2} R_i z^i$, $b^2(z) = \sum_{i=0}^{4} b_i z^i$. We also have the following constraints on the coefficients:*

$$\begin{aligned} na_3 + \frac{n(n-1)}{2}b_4 - R_2 &= 0 \\ (n-1)a_3 + \frac{(n-1)(n-2)}{2}b_4 - R_2 &= 0 \\ na_2 + \frac{n(n-1)}{2}b_3 - R_1 &= 0 \end{aligned}$$

Proof:

Recall the definition of A_i and by Thm 2.2 we have $A_i(z) \in F_n$. Therefore:

$$A_0(z) = -R(z)$$

$$A_1(z) = a(z) - zR(z)$$

$$A_2(z) = 2za(z) + b^2(z) - z^2R(z)$$

Hence the functions $a(z), b^2(z), R(z)$ are all polynomials in z.

On the other hand, we have:

$$A_n(z) = nz^{n-1}a(z) + \frac{n(n-1)}{2}z^{n-2}b^2(z) - z^n R(z) \in F_n$$

Therefore we have:

$$nza(z) + nz^{n-1}a(z) + \frac{n(n-1)}{2}b^2(z) - z^2R(z) \in F_2 \tag{1}$$

Calculating A_{n-1} and A_{n-2} similarly we can get:

$$(n-1)za(z) + \frac{(n-1)(n-2)}{2}b^2(z) - z^2R(z) \in F_3 \tag{2}$$

$$(n-2)za(z) + \frac{(n-2)(n-3)}{2}b^2(z) - z^2R(z) \in F_4 \tag{3}$$

Since F_k is closed under addition and substraction, $(1) - (2)$ gives

$$za(z) + (n-1)b^2(z) \in F_3 \tag{4}$$

$(2) - (3)$ gives

$$za(z) + (n-2)b^2(z) \in F_4 \tag{5}$$

$(4) - (5)$ gives $b^2(z) \in F_4$ and hence $za(z), z^2R(z) \in F_4$. i.e. $a(z) \in F_3$ and $R(z) \in F_2$.

For the relation between the coefficients of functions a, b^2, R, we reconsider

$$\begin{aligned} A_n(z) &= nz^{n-1}a(z) + \frac{n(n-1)}{2}z^{n-2}b^2(z) - z^nR(z) \\ &= nz^{n-1}\left(\sum_{i=0}^{3} a_i z^i\right) + \frac{n(n-1)}{2}z^{n-2}\left(\sum_{i=0}^{4} b_i z^i\right) - z^n\left(\sum_{i=0}^{2} R_i z^i\right) \end{aligned}$$

Since $A_n(z) \in F_n$, the coefficient of term z^{n+2} must vanishes and hence we have:

$$na_3 + \frac{n(n-1)}{2}b_4 - R_2 = 0$$

Similarly by setting the coefficient of z^{n+1} terms of $A_n(z), A_{n-1}(z)$ to be 0, we get the remaining identity on coefficients. □

2.4. **Sufficient Conditions.** The above theorem provide necessary conditions on the functions $a(z), b^2(z), R(z)$ when searching for arbitrage-free polynomial interest rate models. On the other hand, these conditions are also part of sufficient conditions.

Theorem 2.4 (Sufficiency). *Suppose $a(z) = \sum_{i=0}^{3} a_i z^i$, $R(z) = \sum_{i=0}^{2} R_i z^i$, $b^2(z) = \sum_{i=0}^{4} b_i z^i$ are polynomials that satisfy the following conditions for some nature number n:*

$$\begin{aligned} na_3 + \frac{n(n-1)}{2} b_4 - R_2 &= 0 \\ (n-1)a_3 + \frac{(n-1)(n-2)}{2} b_4 - R_2 &= 0 \\ na_2 + \frac{n(n-1)}{2} b_3 - R_1 &= 0 \end{aligned}$$

And the factor process Z_t with $dZ_t = a(Z_t)dt + b(Z_t)dW_t$ is non-explosive. Then there exists a unique model $P_t(T) = \sum_{i=0}^{n} g_i(T-t)Z_t^i$ satisfying condition (A) where the spot interest rates is given by $r_t = R(Z_t)$. Moreover, if $n \leq 3$, the coefficients functions g_i can be calculated explicitly and we have a factor interest rate model with explicit bond price.
If in addition the spot interest rate process $r_t = R(Z_t)$ is non-negative, then the discounted ZCB process $\tilde{P}_t(T)$ is true martingale if and only if the factor process Z_t is bounded.
Proof:

By theorem 2.1, if there exists functions $g_i(x)$ such that the corresponding identity holds with this particular choice of $a(z), b^2(z), R(z)$. Then we can form a polynomial model that satisfies condition (A). Therefore it remains to show that, we can always get a set of unique solution functions $g_i(x)$.

The identity of Theorem 2.1 reads:

$$\sum_{i=0}^{n} \dot{g}_i(x) z^i = \sum_{i=0}^{n} g_i(x) \left((\sum_{i=0}^{3} a_i z^i) \frac{d(z^i)}{dz} + \frac{\sum_{i=0}^{4} b_i z^i}{2} \frac{d^2(z^i)}{dz^2} - (\sum_{i=0}^{2} R_i z^i) z^i \right)$$

Comparing the coefficients of z^{n+2} gives:

$$0 = g_n \left(na_3 + \frac{n(n-1)}{2} b_4 - R_2 \right)$$

Which is checked by the conditions of this theorem. Similarly comparing z^{n+1} terms gives:

$$0 = g_n \left(na_2 + \frac{n(n-1)}{2} b_3 - R_1 \right) + g_{n-1} \left((n-1)a_3 + \frac{(n-1)(n-2)}{2} b_4 - R_2 \right)$$

Which is also checked by the conditions of the theorem. In general, collecting coefficients of z^i terms for $i = 0, 1, \ldots, n$ gives:

$$\dot{g}_i = g_{i+2}\frac{(i+2)(i+1)}{2}b_0 + g_{i+1}\left((i+1)a_0 + \frac{i(i+1)}{2}b_1\right) + g_i\left(ia_1 + \frac{i(i-1)}{2}b_2 - R_0\right)$$
$$(*) \qquad + g_{i-1}\left((i-1)a_2 + \frac{(i-1)(i-2)}{2}b_3 - R_1\right) + g_{i-2}\left((i-2)a_3 + \frac{(i-2)(i-3)}{2}b_4 - R_2\right)$$

The formula (*) holds true provided we interpret terms $g_{-2}, g_{-1}, g_{n+1}, g_{n+2}$ to be 0.

Having the above results, searching for an explicit polynomial interest rate model is equivalent to solve the coupled ODE

$$(**) \qquad \underline{\dot{G}}(x) = S\underline{G}(x)$$

Where $\underline{G}(x) = (g_0(x), \ldots, g_n(x))^T$ and the $(n+1) \times (n+1)$ matrix S is given by:

$$\begin{aligned}
S_{i,i+2} &= \frac{(i+2)(i+1)}{2}b_0 \\
S_{i,i+1} &= (i+1)a_0 + \frac{(i+1)i}{2}b_1 \\
S_{i,i} &= ia_1 + \frac{i(i-1)}{2}b_2 - R_0 \\
S_{i,i-1} &= (i-1)a_2 + \frac{(i-1)(i-2)}{2}b_3 - R_1 \\
S_{i,i-2} &= (i-2)a_3 + \frac{(i-2)(i-3)}{2}b_4 - R_2
\end{aligned}$$

with boundary condition $\underline{G}(0) = (1, 0, \ldots, 0)^T$. We call the matrix S the information matrix because it provides all we need to solve the coupled ODE (**) and hence the identity in Theorem 2.1.

It remains to show that for any information matrix S, the coupled ODE (**) has a unique solution and we can form a unique polynomial model satisfies condition (A). This is true since we always have a unique solution to 1st order ODE with initial conditions. To be more specific, I will show that this is indeed the case for $n = 1$, i.e. the information matrix S is 2×2. For higher dimensions, the proof is similar.

Case 1: suppose S has 2 distinct eigenvalue (maybe complex) λ_i, with corresponding eigenvector $\underline{v}_i$. Then we have the eigenvectors must be linearly independent and the general solution is then $\underline{G}(x) = c_1\underline{v}_1 e^{\lambda_1 x} + c_2\underline{v}_2 e^{\lambda_2 x}$ for some constants c_1, c_2. Applying boundary condition

$\underline{G}(0) = (1, 0, \ldots, 0)^T$, we have $c_1\underline{v}_1 + c_2\underline{v}_2 = (1, 0)^T$. We can then solve for c_1, c_2 and this has a unique solution because vectors $\underline{v}_1, \underline{v}_2$ are linearly independent.

Case 2: S has 2 repeated eigenvalue λ with corresponding eigenvector $\underline{v}_1$. In this case, we can always find a vector $\underline{v}_2$ that is independent of $\underline{v}_1$ and that

$$(S - \lambda I)\underline{v}_2 = \underline{v}_1$$

where I is the identity matrix. In this case, the general solution to (**) is then

$$\underline{G}(x) = c_1\underline{v}_1 e^{\lambda x} + c_2(\underline{v}_2 e^{\lambda x} + \underline{v}_1 x e^{\lambda x})$$

We now apply the boundary condition and still can get a unique solution.

For Higher dimensions, the proof goes similarly. Especially, when $n \leq 3$, the information matrix S is at most 4×4. We know that formula exists for finding roots of quartic equation. Therefore, we can solve (**) explicitly and get a explicit bond price model.

Notice that in general we can write the solution as

$$\underline{G}(x) = e^{Sx}\underline{G}(0)$$

For the rest of this theorem, the discounted ZCB price process $\tilde{P}_t(T)$ given by

$$\tilde{P}_t(T) = \exp\left(-\int_0^t R(Z_s)ds\right) \sum_{i=0}^n g_i(T-t)Z_t^i$$

is bounded if the factor process Z_t is bounded. And we know that bounded local martingales are true martingales.

On the other hand, if $\tilde{P}_t(T)$ are true martingales for any T, then we have

$$\begin{aligned} \tilde{P}_t(T) &= \mathbb{E}[\tilde{P}_T(T)|\mathscr{F}_t] \\ \sum_{i=0}^n g_i(T-t)Z_t^i &= \mathbb{E}\left[\exp\left(-\int_t^T R(Z_s)ds\right)|\mathscr{F}_t\right] \qquad \leq 1 \end{aligned}$$

Hence Z_t must be bounded. $\square$

Remark 2.3. *Theorem 2.4 implies that we can freely choose the factor process and spot rate process as long as the conditions in Theorem 2.4 are satisfied. Then we are guaranteed to get a consistent*

bond price model. Further, for quadratic and cubic models, we can always have an explicit bond price expression.

Remark 2.4. *Notice that the situation here is completely different as in the exponential polynomial case, where Filipovic [3] proves that the maximum degree is 2. Here we can choose the polynomial to an arbitrary large degree so long as the sufficient conditions stated in Theorem 2.4 holds. Actually this is the crucial reason that we can extend the model to the stochastic volatility case as we will see in section 3.*

2.5. **Examples of Parametric Families.** Therefore we can start by choosing some stochastic processes that capture some features of the spot rate and set up the corresponding bond price model depending on spot rate process. Here below are some possible choice for the factor process: $(n = 2)$

Example 3 (Family 1)**.**

$$\begin{aligned} dZ_t &= \alpha Z_t(Z_t - k)dt + \sqrt{\beta Z_t(k - Z_t)}dW_t \\ r_t &= 2\alpha Z_t \end{aligned}$$

with parameter $\alpha, \beta, k > 0$*. The information matrix* S *of this family takes the form:*

$$S = \begin{pmatrix} 0 & 0 & 0 \\ -2\alpha & -k\alpha & k\beta \\ 0 & -\alpha & -2k\alpha - \beta \end{pmatrix}$$

Remark 2.5. *This choice of* Z_t *provides non negative spot rate lives in the interval* $[0, 2k\alpha]$*. Also we can see that the information matrix* S *have an eigenvalue 0, hence we can solve the remaining eigenvalues by solving a quadratic equation easily. However, the process might be absorbed at the boundary in finite time.*

Example 4 (Family 2)**.**

$$\begin{aligned} dZ_t &= \alpha(\beta - Z_t)dt + \sqrt{Z_t^3}dW_t \\ r_t &= Z_t \end{aligned}$$

with parameter $\alpha, \beta > 0$. The information matrix S of this family takes the form:

$$S = \begin{pmatrix} 0 & \alpha\beta & 0 \\ -1 & -\alpha & 2\alpha\beta \\ 0 & -1 & -2\alpha \end{pmatrix}$$

Remark 2.6. *This family provides consistent models with factor process being spot rate process itself. Also the spot rate will be non-negative and fluctuating around some mean spot rate level β. The parameter α measures the strength that pull the spot rate to the mean spot rate β. There is no absorption issue in this family but the spot rate is not bounded above.*

Example 5 (Family 3)**.**

$$\begin{aligned} dZ_t &= \alpha(\beta - Z_t)dt + \sqrt{Z_t(k - Z_t)(l - Z_t)}dW_t \\ r_t &= Z_t \end{aligned}$$

with parameter $\alpha > 0$ and $0 < \beta < k \leq l$. The information matrix S of this family takes the form:

$$S = \begin{pmatrix} 0 & \alpha\beta & 0 \\ -1 & -\alpha & 2\alpha\beta + kl \\ 0 & -1 & -2\alpha - k - l \end{pmatrix}$$

Remark 2.7. *This is a modification of family 2 that provides consistent models with factor process Z_t being spot rate process r_t itself. Also the spot rate will be in the interval $[0, k]$ and fluctuating around some mean spot rate level β. Moreover since the factor process is bounded, the discounted bond price provided by this family will be true martingales.*

3. POLYNOMIAL STOCHASTIC VOLATILITY MODELS

3.1. **Model Setup.** In this section, we would like to extend the models from last section and apply them to the stochastic volatility case. For simplicity in this section, we will set the spot interest rate $r_t = 0$. i.e. there is no discounting in time. This assumption greatly simplify the notation and calculation without affecting the generality of our results provided the spot interest rates are deterministic. We will occasionally mention some changes on the results in the stochastic spot rate case where $r_t = R(Z_t)$.

We model the power option price process $P_t(T, \theta)$ as

$$P_t(T,\theta) = S_t^\theta \left(\sum_{i=0}^{n} k_i(T-t,\theta)\phi_i(Z_t) \right)$$

for some deterministic functions k_i and ϕ_i. Again we assume that the factor process Z_t follows:

$$dZ_t = a(Z_t)dt + b(Z_t).dW_t$$

We also assume that under pricing measure, the stock price process S_t follows:

$$dS_t = S_t\sigma_t.dW_t$$

Similar to the interest rate model discussed above, we assume that the volatility process σ_t also depends on the factor Z_t, i.e.

$$\sigma_t = h(Z_t)$$

for some deterministic function $h(z)$.

Remark 3.1. *If we set $\phi_i(z) = z^i$, we recover the polynomial type model for the stochastic volatility case.*

The boundary condition $P_T(T,\theta) = S_T^\theta$ can be made to hold true by setting $k_0(0,\theta) = 1$, $k_i(0,\theta) = 0 \quad i = 1, 2, \ldots, n$ and $\phi_0(z) = 1$. By applying a similar argument as Remark 2.2, there is no loss of generality by assuming this type of boundary conditions on k_i. Hence here follows, we will always use this type of boundary condition.

3.2. **No Arbitrage Condition.** In the following subsection, we are going to establish the interconnection between functions k_i, ϕ_i, a, b. And prove that under some constraints, Black-Scholes model is the only example.

For consistency, we need:

$$\text{(B) : The processes } P_t(T,\theta) \text{ are local martingale under some EMM } \mathbb{Q} \quad \forall (T,\theta)$$

Remark 3.2. *In the stochastic spot rate case, condition (B) will changed to require the processes $\tilde{P}_t(T,\theta)$ are local martingales, where*

$$\tilde{P}_t(T,\theta) = \left(\exp - \int_0^t R(Z_s)ds\right) P_t(T,\theta)$$

For ease of notation, we will introduce the following functions:

$$A(x,\theta,z) := \sum_{i=0}^{n} \frac{\theta(\theta-1)}{2} k_i(x,\theta)\phi_i(z)$$
$$B(x,\theta,z) := \sum_{i=0}^{n} k_i(x,\theta)\frac{\partial \phi_i(z)}{\partial z}$$
$$C(x,\theta,z) := \sum_{i=0}^{n} \frac{1}{2} k_i(x,\theta)\frac{\partial^2 \phi_i(z)}{\partial z^2}$$
$$D(x,\theta,z) := \sum_{i=0}^{n} \theta k_i(x,\theta)\frac{\partial \phi_i(z)}{\partial z}$$

For no arbitrage, we have:

Theorem 3.1. *Condition (B) holds if and only if*

(()
$$\sum_{i=0}^{n} \dot{k}_i(x,\theta)\phi_i(z) = \|h(z)\|^2 A(x,\theta,z) + a(z)B(x,\theta,z) + \|b(z)\|^2 C(x,\theta,z) + b(z).h(z)D(x,\theta,z)*)$$

holds for all (x,θ,z)

Proof:

Applying Ito's formula, we get:

$$dS_t^\theta = S_t^\theta\left(\theta h(Z_t).dW_t + \frac{1}{2}\theta(\theta-1)\|h(z)\|^2 dt\right)$$
$$d\left(\sum_{i=0}^{n} k_i(T-t,\theta)\phi_i(Z_t)\right) = \sum_{i=0}^{n}\left(-\dot{k}_i\phi_i + ak_i\frac{\partial \phi_i}{\partial z} + \frac{1}{2}k_i\|b\|^2\frac{\partial^2\phi_i}{\partial z^2}\right)dt + k_i\frac{\partial \phi_i}{\partial z}b.dW_t$$

Then we could use the product rule for stochastic process to get:

$$dP_t = S_t^\theta d\left(\sum_{i=0}^{n} k_i(T-t,\theta)\phi_i(Z_t)\right) + \left(\sum_{i=0}^{n} k_i(T-t,\theta)\phi_i(Z_t)\right)dS_t^\theta + d\left(\sum_{i=0}^{n} k_i(T-t,\theta)\phi_i(Z_t)\right)dS_t^\theta$$

Finally we set the coefficient of dt to zero and get the desired identity (*). $\square$

Remark 3.3. *In the stochastic spot rate case, we have an extra $-R(z)\sum_{i=0}^{n} k_i(x,\theta)\phi_i(z)$ term on the RHS of identity (*)*

Now if the functions $k_i(x,\theta)$ and $\phi_i(z)$ are known, we would like to invert Theorem 3.1 and solve for functions $a(z), b(z), h(z)$. This is handled by the following:

Theorem 3.2. *If for arbitrary z, the functions $A(\cdot,\cdot,z)$, $B(\cdot,\cdot,z)$, $C(\cdot,\cdot,z)$ and $D(\cdot,\cdot,z)$ are linearly independent, then the functions $a(z)$, $\|b(z)\|^2$, $\|h(z)\|^2$ and $b(z).h(z)$ are uniquely determined by functions $k_i(x,\theta)$ and $\phi_i(z)$.*
Proof:

Fix any z arbitrarily. There are 4 unknowns, namely $a(z)$, $\|b(z)\|^2$, $\|h(z)\|^2$ and $b(z).h(z)$ in the identity derived in Theorem 3.1. Also by linear independency, we can find 4 sample points (x_i,θ_i) with $i=1,2,3,4$ such that the 4 by 4 matrix with ith row formed by vector

$$(A(x_i,\theta_i,z), B(x_i,\theta_i,z), C(x_i,\theta_i,z), D(x_i,\theta_i,z))$$

is invertible. Hence we can solve for $a(z)$, $\|b(z)\|^2$, $\|h(z)\|^2$ and $b(z).h(z)$ and the solution is unique. Since z is fixed arbitrarily, we get the result. $\square$

Remark 3.4. *In fact as we are going to prove later, in the special case that $\phi_i(z)=z^i$, i.e. in polynomial model, we can say more about the functions $a(z)$, $\|b(z)\|^2$, $\|h(z)\|^2$ and $b(z).h(z)$. Actually they are all polynomials in z.*

Now we define the following functions:

$$\alpha_i(z) := \|h(z)\|^2\,\phi_i(z)$$
$$\beta_i(z) := \frac{\partial\phi_i(z)}{\partial z}\cdot a(z) + \frac{\|b(z)\|^2}{2}\cdot\frac{\partial^2\phi_i(z)}{\partial z^2}$$
$$\gamma_i(z) := \frac{\partial\phi_i(z)}{\partial z}b(z).h(z)$$

Theorem 3.3. *if the functions $\frac{\theta(\theta-1)}{2}k_i(x,\theta)$, $k_i(x,\theta)$, $\theta k_i(x,\theta)$ are linearly independent for $i=0,\dots,n$. Then the functions $\alpha_i(z),\beta_i(z),\gamma_i(z)$ are uniquely determined. Moreover, they are linear combinations of $\phi_0(z),\dots,\phi_n(z)$.*
Proof:

We can rewrite the identity established in theorem 3.1 as:

$$\sum_{i=0}^{n} \dot{k}_i(x,\theta)\phi_i(z) = \sum_{i=0}^{n} \alpha_i(z)\frac{\theta(\theta-1)}{2}k_i(x,\theta) + \beta_i(z)k_i(x,\theta) + \gamma_i(z)\theta k_i(x,\theta)$$

By a similar argument as Theorem 2.2. We have that the unknowns are functions $\alpha_i, \beta_i, \gamma_i$ and there are $N = 3(n+1)$ of them. By linear independency, we can take N sample points (x_j, θ_j) and form an invertible matrix with its rows formed by functions $\frac{\theta(\theta-1)}{2}k_i(x,\theta)$, $k_i(x,\theta)$, $\theta k_i(x,\theta)$ that evaluate at points (x_j, θ_j). Therefore the functions $\alpha_i(z), \beta_i(z), \gamma_i(z)$ are uniquely determined.

Moreover, notice that the matrix we formed above is a constant matrix, especially it is independent of z. Hence, when multiplied by its inverse matrix, we get the solution $\alpha_i, \beta_i, \gamma_i$ are purely linear combinations of $\phi_0(z), \ldots, \phi_n(z)$. □

For polynomial model, we then have the following corollary by setting $\phi_i(z) = z^i$:

Corollary 3.4. *For polynomial models, when the independency holds, we have the functions* $\alpha_i(z), \beta_i(z), \gamma_i(z) \in F_n$.

3.3. **Black-Scholes Model is the Only Example.** We are now ready to prove the rather surprising result.

Theorem 3.5. *With the notation above, in polynomial models with factor process Z_t, i.e.*

$$P_t(T,\theta) = S_t^{\theta} \sum_{i=0}^{n} k_i(T-t,\theta)Z_t^i$$

$$dS_t = S_t\sigma_t.dW_t$$

if the volatility process $\sigma_t = h(Z_t)$ is a function of the factor process and we have the functions $\frac{\theta(\theta-1)}{2}k_i(x,\theta)$, $k_i(x,\theta)$, $\theta k_i(x,\theta)$ are linearly independent for $i = 0, \ldots, n$. Then there is no consistent models except the trivial Black-Scholes model.

Proof:

By corollary 3.4, we have the functions $\alpha_i(z), \beta_i(z), \gamma_i(z)$ are polynomials of degree $\leq n$. Especially we have:

$$\begin{aligned} \alpha_0(z) &= \|h(z)\|^2 \quad \text{hence } \|h(z)\|^2 \text{ is polynomial} \\ \beta_1(z) &= a(z) \quad \text{polynomial} \\ \beta_2(z) &= 2za(z) + \|b(z)\|^2 \quad \text{hence } \|b(z)\|^2 \text{ is polynomial} \\ \gamma_1(z) &= b(z).h(z) \quad \text{polynomial} \end{aligned}$$

At the same time, we have:

$$\alpha_n(z) = z^n \|h(z)\|^2 \quad \text{a polynomial of degree} \leq n$$

But the right hand side cannot have degree $\leq n$ unless the degree of $\|h(z)\|^2$ is 0, i.e. a constant, say σ^2. We can then define the following stochastic process:

$$d\tilde{W}_t = \frac{1}{\sigma}\sigma_t.dW_t$$

Notice that process $\tilde{W}_t$ is a local martingale with quadratic variation

$$d[\tilde{W}_t] = \frac{1}{\sigma^2} \|\sigma_t\|^2 dt = dt$$

By Levy's characterisation of Brownian Motion, we conclude that $\tilde{W}_t$ is also a Brownian Motion and we have $dS_t = S_t \sigma d\tilde{W}_t$. Hence with re-parameterizing the factor process Z_t in terms of the new background Brownian motion $\tilde{W}_t$ we have the resulting process S_t is then geometric Brownian Motion and we recover the Black-Scholes model. □

3.4. **An Extension To Polynomial Model.** In last section, we have seen that under some linearly independency constraints, Black-Scholes model is the only no-arbitrage model that makes power options price process $P_t(T, \theta)$ have the form

$$P_t(T, \theta) = S_t^\theta \sum_{i=0}^{n} k_i(T - t, \theta) Z_t^i \quad \text{for all } \theta$$

Where

$$\begin{aligned} dS_t &= S_t h(Z_t).dW_t \\ dZ_t &= a(Z_t)dt + b(Z_t).dW_t \end{aligned}$$

However, if we first fix θ, and then try to find a no-arbitrage polynomial model. We will be in the similar situation as interest rate case and hence we can have infinitely many examples. To be more specific, we will have infinitely many consistent models of the form

$$P_t(T,\theta) = S_t^\theta \sum_{i=0}^{n(\theta)} k_i(T-t,\theta) Z_t^i(\theta)$$

$$dS_t = S_t h(Z_t(\theta)).dW_t$$

$$dZ_t(\theta) = a(Z_t(\theta),\theta)dt + b(Z_t(\theta),\theta).dW_t$$

That is for each specific value of θ, we choose a specific $n(\theta)$ and a corresponding factor process $Z_t(\theta)$.

So we may wonder can we tweak the original polynomial model such that we allow more no-arbitrage models instead of just Black-Scholes but don't need to choose a brand new factor process Z_t every time we change the value of θ. The answer is yes and we propose a family

$$D_N = \left\{ \frac{i}{N} | i = 1, 2, \ldots, N-1 \right\}$$

When $\theta \in D_N$, we set up our model (M) as

$$P_t(T,\theta) = S_t^\theta \sum_{i=0}^{n(\theta)} k_i(T-t,\theta) Z_t^i(D_N)$$

$$dS_t = S_t h(Z_t(D_N), D_N).dW_t$$

$$dZ_t(D_N) = a(Z_t(D_N), D_N)dt + b(Z_t(D_N), D_N).dW_t$$

i.e. Now the factor process Z_t is family dependent instead of θ dependent. Hence we can use the same factor process Z_t as long as we are in family D_N. For the following, in order to simplify the notation, we will drop D_N dependence on functions a, b, h and still write as $a(z), h(z), b(z)$. But bear in mind that we may choose them to depends on family D_N instead of θ.

Remark 3.5. *If we get a consistent model (M), we then get the exact modelling of price $P_t(T,\theta)$ at $\theta \in D_N$. And hence we are able to approximate the remaining $\theta \in (0,1)$.*

By applying Theorem 3.1, the no-arbitrage condition for our model proposed above is

$$\sum_{i=0}^{n(\theta)} \dot{k}_i(x,\theta)\phi_i(z) = \sum_{i=0}^{n(\theta)} k_i(x,\theta) B_i(z,\theta)$$

Where

$$\begin{aligned} B_i(z,\theta) &= \frac{\theta(\theta-1)}{2}\left\|h(z)\right\|^2\phi_i(z) + d(z,\theta)\frac{\partial\phi_i(z)}{\partial z} + \frac{1}{2}\left\|b(z)\right\|^2\frac{\partial^2\phi_i(z)}{\partial z^2} \\ \phi_i(z) &= z^i \\ d(z,\theta) &= a(z) + \theta b(z).h(z) \end{aligned}$$

By using a similar argument as Theorem 3.3, we have the following

Theorem 3.6. *If for fixed θ, the functions $k_i(\cdot,\theta)$ are linearly independent for $i = 0,1,\ldots,n$. Then $B_i(\cdot,\theta) \in F_{n(\theta)}$*

Remark 3.6. *In the stochastic spot rate case, replace $\frac{\theta(\theta-1)}{2}\left\|h(z)\right\|^2$ by $\frac{\theta(\theta-1)}{2}\left\|h(z)\right\|^2 - R(z)$ in the definition of $B_i(z,\theta)$.*

3.5. **Necessary Conditions and Sufficient Conditions.** Similar to the case of interest rate, we get

Theorem 3.7 (Necessity)**.** *In polynomial stochastic volatility model (M), if for any θ, the coefficient functions $k_i(\cdot,\theta)$ are linearly independent for $i = 0,1,\ldots,n$ then we have $d(z,\theta), \left\|h(z)\right\|^2, \left\|b(z)\right\|^2$ are all polynomials in z and $\left\|h(z)\right\|^2 \in F_2$, $d(z,\theta) \in F_3$, $\left\|b(z)\right\|^2 \in F_4$. Further if $d(z,\theta) = \sum_{i=0}^{3} d_i(\theta)z^i$, $\left\|h(z)\right\|^2 = \sum_{i=0}^{2} h_i z^i$, $\left\|b(z)\right\|^2 = \sum_{i=0}^{4} b_i z^i$. We also have the following constraints on the coefficients:*

$$\begin{aligned} n(\theta)d_3(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_4 + \frac{\theta(\theta-1)}{2}h_2 &= 0 \\ (n(\theta)-1)d_3(\theta) + \frac{(n(\theta)-1)(n(\theta)-2)}{2}b_4 + \frac{\theta(\theta-1)}{2}h_2 &= 0 \\ n(\theta)d_2(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_3 + \frac{\theta(\theta-1)}{2}h_1 &= 0 \end{aligned}$$

Proof:

Recall the definition of B_i and by Thm 3.6 we have $B_i(z,\theta) \in F_{n(\theta)}$ for each θ. Therefore:

$$\begin{aligned} B_0(z,\theta) &= \frac{\theta(\theta-1)}{2}\left\|h(z)\right\|^2 \\ B_1(z,\theta) &= d(z,\theta) + \frac{\theta(\theta-1)}{2}z\left\|h(z)\right\|^2 \\ B_2(z,\theta) &= 2zd(z,\theta) + \left\|b(z)\right\|^2 + \frac{\theta(\theta-1)}{2}z^2\left\|h(z)\right\|^2 \end{aligned}$$

Hence the functions $d(z,\theta), \|h(z)\|^2, \|b(z)\|^2$ are all polynomials in z.

On the other hand, we have:

$$B_{n(\theta)}(z,\theta) = n(\theta)z^{n(\theta)-1}d(z,\theta) + \frac{n(\theta)(n(\theta)-1)}{2}z^{n(\theta)-2}\|b(z)\|^2 + z^{n(\theta)}\frac{\theta(\theta-1)}{2}\|h(z)\|^2 \in F_{n(\theta)}$$

Therefore we have:

$$n(\theta)zd(z,\theta) + \frac{n(\theta)(n(\theta)-1)}{2}\|b(z)\|^2 + z^2\frac{\theta(\theta-1)}{2}\|h(z)\|^2 \in F_2 \tag{6}$$

Calculating $B_{n(\theta)-1}$ and $B_{n(\theta)-2}$ similarly we can get:

$$(n(\theta)-1)zd(z,\theta) + \frac{(n(\theta)-2)(n(\theta)-1)}{2}\|b(z)\|^2 + z^2\frac{\theta(\theta-1)}{2}\|h(z)\|^2 \in F_3 \tag{7}$$

$$(n(\theta)-2)zd(z,\theta) + \frac{(n(\theta)-2)(n(\theta)-3)}{2}\|b(z)\|^2 + z^2\frac{\theta(\theta-1)}{2}\|h(z)\|^2 \in F_4 \tag{8}$$

Since F_k is closed under addition and substraction, (6) $-$ (7) gives

$$zd(z,\theta) + (n(\theta)-1)\|b(z)\|^2 \in F_3 \tag{9}$$

(7) $-$ (8) gives

$$zd(z,\theta) + (n(\theta)-2)\|b(z)\|^2 \in F_4 \tag{10}$$

(9) $-$ (10) gives $\|b(z)\|^2 \in F_4$ and hence $zd(z,\theta), \frac{\theta(\theta-1)}{2}\|h(z)\|^2 \in F_4$. i.e. $d(z,\theta) \in F_3$ and $\|h(z)\|^2 \in F_2$.

For the relation between the coefficients of functions $d(z,\theta), \|h(z)\|^2, \|b(z)\|^2$, we reconsider

$$\begin{aligned} B_{n(\theta)}(z,\theta) &= n(\theta)z^{n(\theta)-1}d(z,\theta) + \frac{n(\theta)(n(\theta)-1)}{2}z^{n(\theta)-2}\|b(z)\|^2 + z^{n(\theta)}\frac{\theta(\theta-1)}{2}\|h(z)\|^2 \\ &= n(\theta)z^{n(\theta)-1}\left(\sum_{i=0}^{3} d_i(\theta)z^i\right) + \frac{n(\theta)(n(\theta)-1)}{2}z^{n(\theta)-2}\left(\sum_{i=0}^{4} b_i z^i\right) + z^{n(\theta)}\frac{\theta(\theta-1)}{2}\left(\sum_{i=0}^{2} h_i z^i\right) \end{aligned}$$

Since $B_{n(\theta)}(z,\theta) \in F_{n(\theta)}$, the coefficient of term $z^{n(\theta)+2}$ must vanishes and hence we have:

$$n(\theta)d_3(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_4 + \frac{\theta(\theta-1)}{2}h_2 = 0$$

Similarly by setting the coefficient of $z^{n(\theta)+1}$ terms of $B_{n(\theta)}(z,\theta), B_{n(\theta)-1}(z,\theta)$ to be 0, we get the remaining identity on coefficients. □

Likewise with the same argument as theorem 2.4, we have

Theorem 3.8 (Sufficiency). *Suppose $d(z,\theta) = \sum_{i=0}^{3} d_i(\theta)z^i$, $\|h(z)\|^2 = \sum_{i=0}^{2} h_i z^i$, $\|b(z)\|^2 = \sum_{i=0}^{4} b_i z^i$ are polynomials in z that satisfy the following conditions for some nature number $n(\theta)$:*

$$
\begin{aligned}
n(\theta)d_3(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_4 + \frac{\theta(\theta-1)}{2}h_2 &= 0 \\
(n(\theta)-1)d_3(\theta) + \frac{(n(\theta)-1)(n(\theta)-2)}{2}b_4 + \frac{\theta(\theta-1)}{2}h_2 &= 0 \\
n(\theta)d_2(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_3 + \frac{\theta(\theta-1)}{2}h_1 &= 0
\end{aligned}
$$

And the factor process Z_t with $dZ_t = a(Z_t)dt + b(Z_t).dW_t$ is non-explosive. Then there exists a unique model $P_t(T,\theta) = S_t^\theta \sum_{i=0}^{n(\theta)} k_i(T-t)Z_t^i$ satisfying condition (B) where the volatility process is given by $\sigma_t = h(Z_t)$. Moreover, in the case $n(\theta) \leq 3$, the coefficients functions k_i can be calculated explicitly.

3.6. **Worked Examples.** We will discuss some specific examples of model (M) in this subsection. We have already seen in section 3.3 that if $\|h(z)\|^2$ is a constant, .i.e. polynomial with degree 0. Then we will recover the Black-Scholes model. Hence in order to find interesting consistent examples, we must have $\|h(z)\|^2$ at least a linear function, say $\|h(z)\|^2 = h_1 z + h_0$. By Theorem 3.7, this corresponds to the case $h_2 = 0$ and hence we must have $d_3(\theta) = 0$ and $b_4 = 0$. Therefore by Theorem 3.8, the sufficient conditions becomes the factor process Z_t is non-explosive and

$$
n(\theta)d_2(\theta) + \frac{n(\theta)(n(\theta)-1)}{2}b_3 + \frac{\theta(\theta-1)}{2}h_1 = 0 \tag{11}
$$

Given that $\theta \in D_N$, we must find suitable functions a, b, h that depend on N instead of depend on each single θ and nature number $n(\theta)$ that satisfy (11).

Example 6 (Family 4).

$$
\begin{aligned}
\|h(z)\|^2 &= 2N^2 z + h_0 \\
\|b(z)\|^2 &= b_2 z^2 + b_1 z + b_0 \\
b(z).h(z) &= 0 \\
a(z) &= z^2 + d_1 z + d_0 \\
n\left(\frac{i}{N}\right) &= i(N-i)
\end{aligned}
$$

Where b_i, d_i, h_0 are arbitrary constant that produce non-negative and non-explosive factor process Z_t.

Proof:

Since $b(z).h(z) = 0$, we have $d(z, \theta) = a(z) = z^2 + d_1 z + d_0$ and hence function d depends on family D_N only with $d_2 = 0$. We can simply check that (11) holds true and by sufficient theorem 3.8. No-arbitrage model is guaranteed. □

Remark 3.7. *Notice that in above example, the functions $a(z), b(z)$ are actually independent of family D_N. Hence the advantage is that we can use the same factor process Z_t for any precision N. However the drawback is that $n(\theta)$ can be as large as $\frac{N^2}{4}$, which means the corresponding information matrix S can be large. i.e. $\frac{N^2}{4} \times \frac{N^2}{4}$. Therefore solving the eigenvalues and eigenvectors of S is not easy.*

Remark 3.8. *For the purpose of the non-negative and non-explosive factor process Z_t, we can choose from the following parametric family*

$$dZ_t = (Z_t - \alpha_1)(Z_t - \alpha_2)dt + \sqrt{\beta Z_t(\gamma - Z_t)}dW_t^1$$

Where the parameters are $\beta > 0$ and $0 < \alpha_1 < \gamma < \alpha_2$. This parametric family will produce factor process $Z_t \in [0, \gamma]$ without any absorbing at boundary issue.

Example 7 (Family 5)**.**

$$\begin{aligned}
\|h(z)\|^2 &= cN^2 z + h_0 \\
\|b(z)\|^2 &= -cz^3 + b_2 z^2 + b_1 z + b_0 \\
b(z).h(z) &= 0 \\
a(z) &= \frac{N-1}{2} cz^2 + d_1 z + d_0 \\
n(\frac{i}{N}) &= \min\{i, N - i\}
\end{aligned}$$

Where $c > 0$ and b_i, d_i, h_0 are arbitrary constant that produce non-negative and non-explosive factor process Z_t.

Proof:

Simply check condition (11) and apply sufficiency Theorem. □

Remark 3.9. *Since the linear coefficient of* $\|h(z)\|^2$ *is* cN^2*, for any prescribed precision* N*, we could still choose* $c > 0$ *freely so that the norm of volatility process* $\|\sigma_t\|^2 = \|h(Z_t)\|^2$ *can be arbitrary linear function with positive linear term. Moreover we see that* $n(\theta)$ *is at most* $\frac{N}{2}$*, the complexity of the information matrix* S *is reduced significantly. i.e. at most* $\frac{N}{2} \times \frac{N}{2}$*.*

Remark 3.10. *For the purpose of the non-negative and non-explosive factor process* Z_t*, we can choose from the following parametric family*

$$dZ_t = \frac{N-1}{2} c(Z_t - \alpha_1)(Z_t - \alpha_2)dt + \sqrt{cZ_t(Z_t + \beta)(\gamma - Z_t)} dW_t^1$$

Where the parameters are $\beta \geq 0$ *and* $0 < \alpha_1 < \gamma < \alpha_2$*. This parametric family will produce factor process* $Z_t \in [0, \gamma]$ *without any absorbing at boundary issue.*

4. FURTHER ANALYSIS AND EXAMPLES

In this section, we work through some examples that we purposed earlier with simulated graphic results.

4.1. **Stationary Distribution.** To find the stationary distribution for a diffusion process Z_t such that

$$dZ_t = a(Z_t)dt + b(Z_t)dW_t$$

We write $p(t,x,y)$ be the transition density of process Z_t in time t and let

$$f(t,y) = \int_{-\infty}^{\infty} g(x)p(t,x,y)dx$$

be the density at time t when the initial r_0 have density $g(x)$. Then we know that f satisfies the forward equation (Fokker-Planck equation).

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}(fa) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(fb^2)$$

If the initial density $g(x)$ is stationary, then all following density will stay the same, i.e.

$$f(t,y) = g(y) \quad \forall t$$

Hence the stationary distribution g must satisfy the equation

$$-\frac{\partial}{\partial y}(ga) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(gb^2) = 0$$

Solve the above equation gives

$$g(y) \propto \frac{1}{b(y)}\left(\exp\int \frac{2a}{b^2}\right) \tag{12}$$

4.2. **Example on Interest Rate Case.** Here we would like to have a close look at the Family 2 that we purposed in section 2.5. For a recap, we construct a family of quadratic bond price models that depends on the spot rate process r_t and takes the form:

$$P_t(T) = g_0(T-t) + g_1(T-t)r_t + g_2(T-t)r_t^2$$

Where the spot rate process r_t comes from the family

$$dr_t = \alpha(\beta - r_t)dt + \sqrt{r_t^3}dW_t$$

for some positive parameter α, β.

First notice that this family has an stationary distribution that can be calculated explicitly. By formula (12) with $a(y) = \alpha(\beta - y)$ and $b(y) = \sqrt{y^3}$, we get that the stationary distribution R have a density function

$$g_R(r) \propto \frac{1}{r^3} \exp\left(\frac{2\alpha}{r} - \frac{\alpha\beta}{r^2}\right) \quad r > 0$$

Which doesn't look very nice currently. A change of variable $X = 1/R$ reveals the density funtion of X is

$$\begin{aligned} f_X(x) &= g_R(1/x)\left|\frac{dr}{dx}\right| \\ &\propto x \exp\left(-\alpha\beta(x - \frac{1}{\beta})^2\right) \quad x > 0 \end{aligned}$$

To find the normalising constant of density f_X, we integrate over $[0, \infty)$ and set the result to 1 gives

$$\begin{aligned} f_X(x) &= Cx \exp\left(-\alpha\beta(x - \frac{1}{\beta})^2\right) \quad x > 0 \\ C &= \left(\frac{1}{2\alpha\beta} e^{-\frac{\alpha}{\beta}} + 2\sqrt{\frac{\pi\alpha}{\beta}}\Phi(\sqrt{\frac{2\alpha}{\beta}})\right)^{-1} \end{aligned}$$

Where Φ denotes the cumulative standard normal distribution function. Hence the CDF of the stationary distribution is given explicitly by

$$\begin{aligned} \mathbb{P}[R \le r] &= \mathbb{P}[X \ge \frac{1}{r}] \\ &= \int_{1/r}^{\infty} Cx \exp\left(-\alpha\beta(x - \frac{1}{\beta})^2\right) dx \\ &= C\left(\frac{1}{2\alpha\beta} \exp\{-\alpha\beta(\frac{1}{r} - \frac{1}{\beta})^2\} + 2\sqrt{\frac{\pi\alpha}{\beta}}\Phi(\sqrt{2\alpha\beta}(\frac{1}{r} - \frac{1}{\beta}))\right) \quad r > 0 \end{aligned}$$

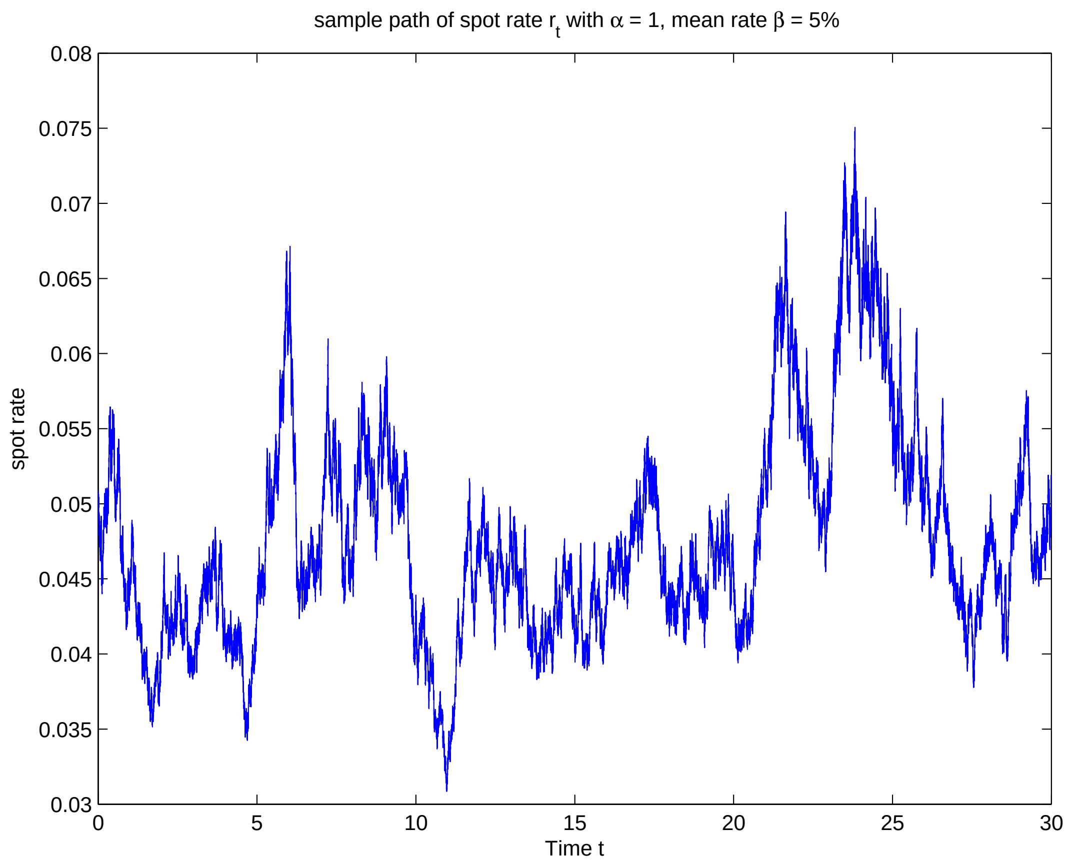
To solve the model completely, we need the information matrix S, in our case with above parameter, we have

$$S = \begin{pmatrix} 0 & 1/20 & 0 \\ -1 & -1 & 1/10 \\ 0 & -1 & -2 \end{pmatrix}$$

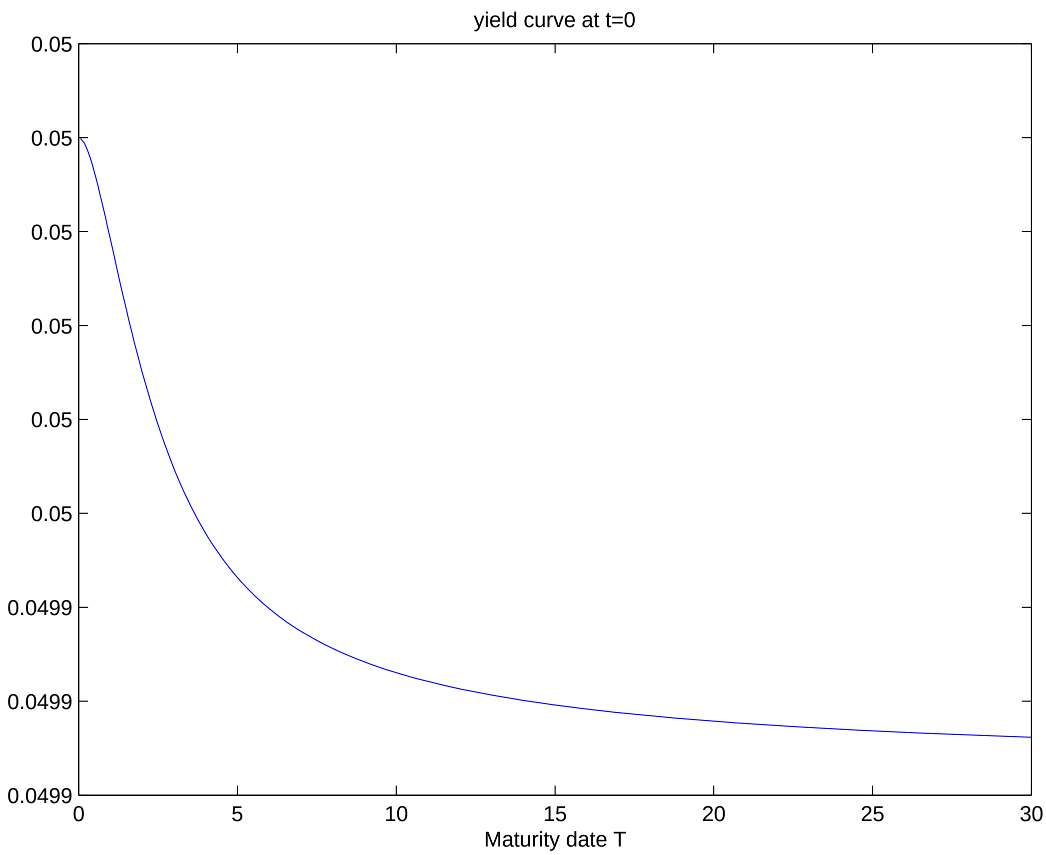
Solve the coefficient functions g_i and the corresponding yield curve is given by

$$Y_t(T) = -\frac{\log P_t(T)}{T-t}$$

Here below, we provide simulated sample path of spot rate process r_t and yield curve $Y_0(T)$ of Family 2, with parameter $\alpha = 1$ and $\beta = 5\%$. Recall that β can be interpret as the mean rate and we can see the path fluctuate between 0.05 and stays positive as predicted by Remark 2.6.



sample path of spot rate r_t with α = 1, mean rate β = 5%
spot rate
Time t
0.08
0.075
0.07
0.065
0.06
0.055
0.05
0.045
0.04
0.035
0.03
0
5
10
15
20
25
30


yield curve at t=0
0.05
0.05
0.05
0.05
0.05
0.05
0.0499
0.0499
0.0499
0
5
10
15
20
25
30
Maturity date T

4.3. **Sample Path on Stochastic Volatility Case.** Here we simulate sample path of Example 7 in section 3.6. We choose our parameters to be $N = 100$, $c = 10^{-4}$, $h_0 = 0$, $\alpha_1 = 0.02$, $\alpha_1 = 0.06$, $\beta = 0$ and $\gamma = 0.05$.

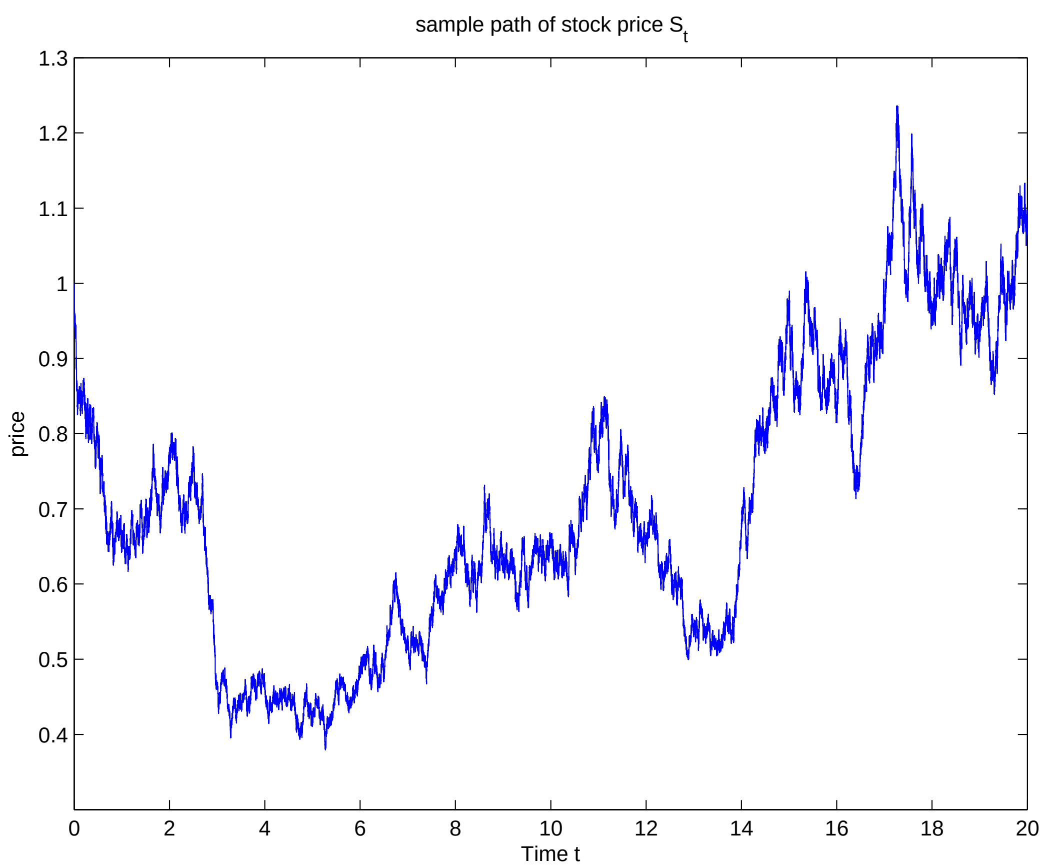
Hence our model reads

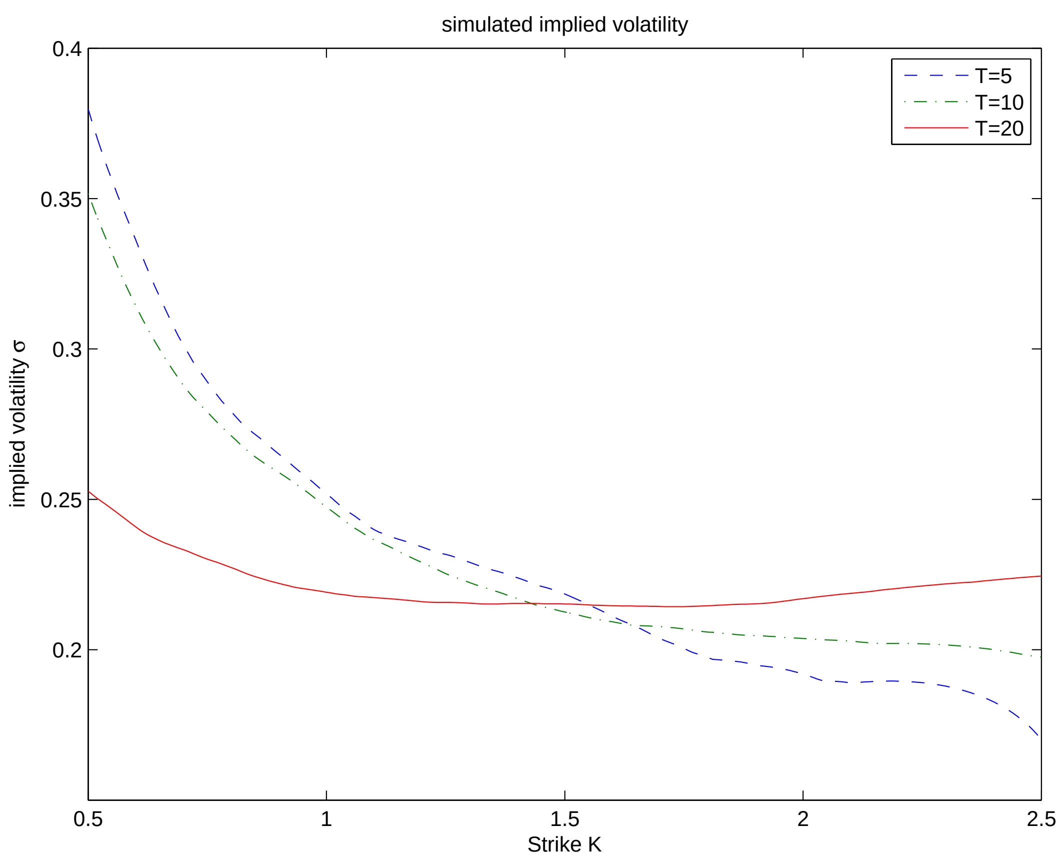
$$dZ_t = \frac{99 \times 10^{-4}}{2}(Z_t - 0.02)(Z_t - 0.06)dt + 0.01 \times \sqrt{Z_t^2(0.05 - Z_t)}dW_t^1$$
$$dS_t = S_t\sqrt{Z_t}dW_t^2$$

Where W^1, W^2 are independent Brownian motions.

Remark 4.1. *By Remark 3.7, we know that factor process Z_t will live inside $[0, 0.05]$, which tells us that the volatility process $\sigma_t = h(Z_t) = \sqrt{Z_t}$ is around 20%.*

Here below, we provide a graph showing the sample path of stock price S_t with initial $S_0 = 1$ and a graph showing the implied volatility with maturity date $T = 5, 10, 20$ and strike K ranges from 0.5 to 2.5.


sample path of stock price S_t
price
Time t
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
2
4
6
8
10
12
14
16
18
20


simulated implied volatility
T=5
T=10
T=20
implied volatility σ
Strike K
0.4
0.35
0.3
0.25
0.2
0.5
1
1.5
2
2.5

5. Appendix A

Here we will define the forward variance model and prove that under some linear independency conditions, if the forward variance process is polynomial function of factor process Z_t, then the maximum degree is 2.

We work under the pricing EMM $\mathbb{Q}$ and model the stock price process by $dS_t = S_t\sigma_t.dW_t$. We parameterize the process $\{P_t(T,\theta)\}$ by

$$P_t(T,\theta) = S_t^{\theta} e^{-\frac{1}{2}\theta(1-\theta)\int_t^T f(t,s;\theta)ds}$$

Where $df(t,T;\theta) = B(t,T;\theta).dW_t + A(t,T;\theta)dt$

Theorem 5.1 (No arbitrage conditions)**.** *If we have*

$$A(t,T;\theta) = B(t,T;\theta).(\frac{\theta(1-\theta)}{2}\int_t^T B(t,s;\theta)ds - \theta\sigma_t) \tag{13}$$

and the process $f(t,T;\theta)$ satisfies

$$f(t,t;\theta) = \|\sigma_t\|^2 \quad \forall(t,\theta) \tag{14}$$

Then the price process of power options $P_t(T,\theta)$ are local martingales for all (T,θ).
Proof:

Let $M_t = \theta\log S_t - \frac{1}{2}\theta(1-\theta)\int_t^T f(t,s;\theta)$. Then $P_t(T,\theta) = e^{M_t}$ is a local martingale if and only if the drift term of $dM_t + \frac{1}{2}d[M]_t$ vanishes, where $[M]$ denotes the quadratic variation of M. By applying Ito's formula, we have:

$$\begin{aligned} dM_t &= \theta\frac{1}{S_t}dS_t - \frac{\theta}{2S_t^2}d[S]_t + \frac{1}{2}\theta(1-\theta)f(t,t;\theta)dt - \frac{1}{2}\theta(1-\theta)\int_t^T df(t,s;\theta)ds \\ &= (\theta\sigma_t - \frac{1}{2}\theta(1-\theta)\int_t^T B(t,s;\theta)ds).dW_t - (\frac{1}{2}\theta(1-\theta)\int_t^T A(t,s;\theta)ds + \frac{1}{2}\theta^2\|\sigma_t\|^2)dt \end{aligned}$$

$$d[M]_t = \left(\theta^2\|\sigma_t\|^2 - \theta^2(1-\theta)\sigma_t.\int_t^T B(t,s;\theta)ds + \frac{1}{4}\theta^2(1-\theta)^2\left\|\int_t^T B(t,s;\theta)ds\right\|^2\right)dt$$

Hence the dt term of $dM_t + \frac{1}{2}d[M]_t$ is

$$-\frac{1}{2}\theta(1-\theta)\int_t^T A(t,s;\theta)ds - \frac{1}{2}\theta^2(1-\theta)\sigma_t.\int_t^T B(t,s;\theta)ds + \frac{1}{8}\theta^2(1-\theta)^2\left\|\int_t^T B(t,s;\theta)ds\right\|^2$$
$$= -\frac{1}{4}\theta^2(1-\theta)^2\int_t^T B(t,s;\theta).\int_t^s B(t,u;\theta)duds + \frac{1}{8}\theta^2(1-\theta)^2\left\|\int_t^T B(t,s;\theta)ds\right\|^2$$

which vanishes by symmetry. □

Remark 5.1. *Changing into Musiela [6] notation, i.e. Setting $f_t(x,\theta) = f(t,t+x;\theta)$ we can recover the model we introduced in Section 1.*

For general factor models, if we require the forward variance to have the form

$$f_t(x,\theta) = f(t,t+x;\theta) = g(x,\theta,Z_t)$$

Hence

$$f(t,T;\theta) = g(T-t,\theta,Z_t)$$

Where the factor process Z_t follows

$$dZ_t = a(Z_t)dt + b(Z_t).dW_t$$

for some continuous deterministic functions $a(z), b(z), g(x,\theta,z)$. Moreover, if we insists the process σ_t depends also on the factor process Z_t, say

$$\sigma_t = h(Z_t)$$

Then by applying Theorem 5.1, we can get the no arbitrage conditions for general factor model.

Theorem 5.2 (No arbitrage conditions for general factor models)**.** *Suppose the functions $a(z)$, $b(z)$, $g(x,\theta,z)$, $h(z)$ satisfy the following*

$$g_x = \frac{1}{2}\|b\|^2\left(g_{zz} - \theta(1-\theta)g_z\int_0^x g_z(y,\theta,z)dz\right) + (\theta b.h + a)g_z$$
$$g(0,\theta,z) = \|h(z)\|^2 \quad \forall(\theta,z)$$

Then we have an arbitrage-free factor model for forward variance by setting:

$$f(t,T;\theta) = g(T-t,\theta,Z_t)$$
$$\sigma_t = h(Z_t)$$
$$dZ_t = a(Z_t)dt + b(Z_t).dW_t$$

Proof:

By applying Theorem 5.1, we have

$$g(0,\theta,Z_t) = f(t,t;\theta) = \|\sigma_t\|^2 = \|h(Z_t)\|^2$$

Applying Ito's formula on $f(t,T;\theta)$ gives:

$$\begin{aligned} df(t,T;\theta) &= -g_x dt + g_z dZ_t + \frac{1}{2} g_{zz} d[Z]_t \\ &= bg_z.dW_t + (ag_z + \frac{1}{2}\|b\|^2 g_{zz} - g_x)dt \quad \text{evaluated at point } (T-t,\theta,Z_t) \end{aligned} \tag{15}$$

Hence by reading off the coefficient dW_t term, we must have $B(t,T;\theta) = b(Z_t)g_z(T-t,\theta,Z_t)$ and by (13), we have:

$$\begin{aligned} A(t,T;\theta) &= g_z b. \left(\frac{theta(1-\theta)}{2} \int_t^T g_z(s-t,\theta,Z_t)b(Z_t)ds - \theta h(Z_t) \right) \\ &= \|b\|^2 \frac{\theta(1-\theta)}{2} g_z \int_0^{T-t} g_z(x,\theta,Z_t)dx - \theta g_z h.b \end{aligned} \tag{16}$$

Equating (15) and (16) gives the required results. □

Integrate the no arbitrage condition of Theorem 5.2 with respect to the first argument from 0 to x. Writing

$$G(x,\theta,z) = \int_0^x g(y,\theta,z)dy$$

We have:

$$G_x = \frac{1}{2}\|b\|^2 (G_{zz} - \frac{\theta(1-\theta)}{2} G_z^2) + (\theta b.h + a)g_z + \|h(z)\|^2 \tag{17}$$

$$G(0,\theta,z) = 0 \quad \forall z \tag{18}$$

For polynomial forward variance models, $G(x,\theta,z) = \sum_{i=0}^{n} k_i(x,\theta) z^i$ for some deterministic functions k_i. Then (17) takes the form:

Theorem 5.3. *We can generate an arbitrage-free polynomial forward variance model if*

$$\sum_{i=0}^{n} \frac{\partial k_i}{\partial x}(x,\theta) z^i = \frac{1}{2} \|b(z)\|^2 \left(\sum_{i=0}^{n} k_i(x,\theta) \frac{\partial^2 z^i}{\partial z^2} - \frac{\theta(1-\theta)}{2} (\sum_{i=0}^{n} k_i(x,\theta) \frac{\partial z^i}{\partial z})^2 \right)$$

$$+ (\theta b(z).h(z) + a(z))(\sum_{i=0}^{n} k_i(x,\theta) \frac{\partial z^i}{\partial z}) + \|h(z)\|^2 \tag{19}$$

holds for all (x,θ,z) and $k_i(0,\theta) = 0$ for all θ and i

Define

$$\alpha_{ij}(z) = \alpha_{ji}(z) := \frac{1}{4} \|b(z)\|^2 \frac{\partial z^i}{\partial z} \frac{\partial z^j}{\partial z}$$

$$\beta_i(z) := \frac{1}{2} \|b(z)\|^2 \frac{\partial^2 z^i}{\partial z^2} + a(z) \frac{\partial z^i}{\partial z}$$

$$\gamma_i(z) := b(z).h(z) \frac{\partial z^i}{\partial z}$$

We can rewrite (19) as

$$\sum_{i=0}^{n} \frac{\partial k_i}{\partial x}(x,\theta) z^i = \sum_{i=0}^{n} k_i(x,\theta) \beta_i(z) + \sum_{i=0}^{n} \theta k_i(x,\theta) \gamma_i(z)$$

$$- \sum_{i,j=0}^{n} \theta(1-\theta) k_i(x,\theta) k_j(x,\theta) \alpha_{ij}(z) + \|h(z)\|^2 \tag{20}$$

Theorem 5.4. *If the functions $1, k_i(x,\theta), \theta k_i(x,\theta), \theta(1-\theta) k_i(x,\theta) k_j(x,\theta)$, $0 \le i \le j \le n$, are linearly independent. Then $\alpha_{ij}, \beta_i, \gamma_i, \|h\|^2$ are uniquely determined and are polynomials of z with degree $\le n$*

Proof:

Set $C = 1 + (n+1) + (n+1) + \frac{(n+2)(n+1)}{2}$ be the total number of unknown functions $\alpha_{ij}(z)$, $\beta_i(z)$, $\gamma_i(z)$, $\|h(z)\|^2$. Similar to the proof of Theorem 2.2, we can take C sample point (x_m, θ_m) for $m = 1, 2, \ldots, C$ and form an invertible $C \times C$ matrix whose m-th row consists of $1, k_i(x_m,\theta_m), \theta_m k_i(x_m,\theta_m)$, $\theta_m(1-\theta_m) k_i(x_m,\theta_m) k_j(x_m,\theta_m)$. Hence the solution is unique.

In addition, we have the matrix we formed above is independent of z. Therefore its inverse is also independent of z and the solutions must be linear combination of $z^0, \ldots, z^n$. $\square$

We now prove our main result on polynomial forward variance model which is similar as in the interest model case [3]:

Theorem 5.5 (Maximal Degree in Polynomial Forward Variance Model)**.** *Given any polynomial model, if the functions* $1, k_i(x,\theta), \theta k_i(x,\theta), \theta(1-\theta)k_i(x,\theta)k_j(x,\theta)$, $1 \leq i \leq j \leq n$, *are linearly independent. Then there is no arbitrage-free model for* $n \geq 3$

Proof:

Recall the definition of α, we have:

$$\begin{aligned}\alpha_{1,1}(z) &= \frac{1}{4}\left\|b(z)\right\|^2 \frac{\partial z}{\partial z}\frac{\partial z}{\partial z} \\ &= \frac{1}{4}\left\|b(z)\right\|^2 \\ \alpha_{nn}(z) &= \frac{1}{4}\left\|b(z)\right\|^2 \frac{\partial z^n}{\partial z}\frac{\partial z^n}{\partial z} g \\ &= \frac{1}{4}\left\|b(z)\right\|^2 n^2 z^{2n-2} \qquad (*)\end{aligned}$$

But by Theorem 5.4 we have both $\alpha_{nn}(z)$ and $\left\|b(z)\right\|^2$ are polynomial with degree at most n. But the right hand side of (*) cannot have degree at most n unless $n \leq 2$. $\square$

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